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Do the Simple and 2/3 Majority Models Belong to the Same Universality Class?: A Monte Carlo Approach

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We extend the majority model (introduced by Tsallis in 1982) in the sense that the required majority might be different from the simple majority. We simulate these models for typical cases which include simple and 2/3 majorities. We exhibit the average cluster size as well as the order parameter as functions of p, the concentration of one of the two possible constituents. No crossover exists between the simple- and non-simple-majority models.

KEY WORDS: Geometrical critical phenomena; majority model; crossover; Monte Carlo.

Majority-rules arguments have been frequently used in connection with real-space renormalization groups (RG). Along this line a specific geometrical model *(majority model)* was introduced in 1982.[1] The model, which refers to plaquettes of *two* possible colors *(bichromatic model)* and which we recall in detail later, was on that occasion discussed within an RG framework. More precisely, its finite-size scaling properties were analytically exhibited and compared with similar suggestions[2,3] for standard models. Within the same philosophy, the model was recently generalized[4] in order to permit an *arbitrary* number of different colors *(polychromatic model)*. However, even in its generalized version, the majority which the model is based upon is the *simple majority*. In other words, and assuming we illustrate the concept for the bichromatic model,

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majority occurs when more than 1/2 of the plaquettes belong to a given color. It is along this line that we shall here generalize the model. More specifically, in the present paper we consider majorities *higher* than the simple one, i.e., majority occurs (in the bichromatic model) when more than p_0 of the plaquettes belong to a given color ($p_0 = 1/2$ and $p_0 = 2/3$, respectively, correspond to simple and 2/3 majorities). Within a Monte Carlo-like simulation framework, we study the cases $1/2 \leqslant p_0 \leqslant 1$, with emphasis on the problem of whether $p_0 - 1/2$ is or is not a relevant parameter for the critical universality class of the system.

We consider, for a given choice of p_0, a (periodic) one-dimensional L-sized strip of "black" and "white" randomly chosen plaquettes; the strip has been constructed with occupancy probabilities for the black and white colors respectively given by p and $1-p$ with $0 \leqslant p \leqslant 1$ (in fact, the dimensionality of the system is completely irrelevant[1,4]; furthermore, the system could as well be constructed on an arbitrary set, not necessarily a Bravais lattice). We then choose randomly (or through any other manner) a plaquette (called *center*) of the strip, and start checking what color is the majority for that particular strip configuration (obtained with a fixed value of p): for simplicity, we check by symmetrically enlarging the piece of strip around the center (i.e., by considering strip clusters of length 1, 3, 5,..., up to approximately L). We call l the length of the strip cluster for which a change of majority occurs (from say *black majority* into white majority) if it does; if the randomly chosen center turned out to be white, then $l = 0$ for this event. We repeat this operation N_0 times (each of which corresponds to a new random strip configuration). We calculate the *black-dominating* mean cluster size $\langle l \rangle_{\text{all clusters}}$ by taking into account *all* clusters as well as the mean finite cluster size $\langle l \rangle_{\text{finite clusters}} \equiv \xi$ by only considering "finite" clusters (to be more precise, by only considering those clusters whose majority has changed *before* l equals L; if no change has occurred up to size L, the cluster will be referred to as "infinite," a denomination which of course becomes strictly correct only in the $L \to \infty$ limit). It is clear that in the $(L, N_0) \to (\infty, \infty)$ limit: (i) when p increases from 0 to p_0, $\langle l \rangle_{\text{all clusters}}$ and ξ coincide and vary from 0 to infinity; (ii) when p increases from p_0 to 1, $\langle l \rangle_{\text{all clusters}}$ remains infinite, whereas ξ decreases from infinity to 0; (iii) when p decreases from 1 to $1 - p_0$, $\langle l \rangle_{\text{all clusters}}$ remains infinite, whereas ξ increases from 0 to infinity; (iv) finally, when p decreases from $1 - p_0$ to 0, $\langle l \rangle_{\text{all clusters}}$ and ξ coincide and decrease from infinity to 0. In the $(L, N_0) \to (\infty, \infty)$ limit, we expect ξ to diverge as $|p - p_c|^{-\nu}$ in the neighborhood of the critical point ($p_c = p_0$ when p increases from 0 to 1, and $p_c = 1 - p_0$ when p decreases from 1 to 0). To be more precise, we expect, for increasing p, $\xi \sim A_{\uparrow}^{-}(p_0 - p)^{-\nu}$ for $p < p_0$ and $\xi \sim A_{\uparrow}^{+}(p - p_0)^{-\nu}$ for $p > p_0$; for decreasing p we expect $\xi \sim A_{\downarrow}^{+}[p - (1 - p_0)]^{-\nu}$ for

$p > (1 - p_0)$ and $\xi \sim A_{\downarrow}^{-}[(1 - p_0) - p]^{-\nu}$ for $p < (1 - p_0)$. The results corresponding to $p_0 = 1/2$ and $p_0 = 2/3$ are indicated in Figs. 1 and 2. Let us stress that Fig. 1b is not symmetric with respect to the $p = 1/2$ axis; indeed, we are all the time focusing on the black clusters and black majority is first attained, for increasing p, at $p = 2/3$, whereas it is lost, for decreasing p, at $p = 1/3$. In Fig. 3a, the p_0 evolution of $A_{\uparrow}^{+}$, $A_{\uparrow}^{-}$, $A_{\downarrow}^{+}$, and $A_{\downarrow}^{-}$ is indicated as well. The results indicated in these figures have been obtained with typical values of $N_0 = 20{,}000$ and $L = 50{,}000$. However, as we approach the critical point p_c, we must use increasingly larger values of L in order to avoid finite-size effects, as well as increasingly larger values of N_0 in order to have sufficiently small statistical fluctuations. As illustrated in Fig. 2, ν seems to be p_0 independent and equals 1 ± 0.1 (hence, for d-dimensional Bravais lattices we have $\nu d = 1 \pm 0.1$). In other words, since no crossover is observed, $p_0 - 1/2$ is an irrelevant parameter as far as criticality is concerned. The RG results for the simple-majority model[1,4] yield $\nu d = 2$, i.e., the average cluster size discussed within RG appears to be the square of ξ introduced here. Why this should be so remains an open question.

Let us now focus on the *order parameter* m defined (in analogy with the *percolation* order parameter; see, for instance, ref. 5) as follows. Once

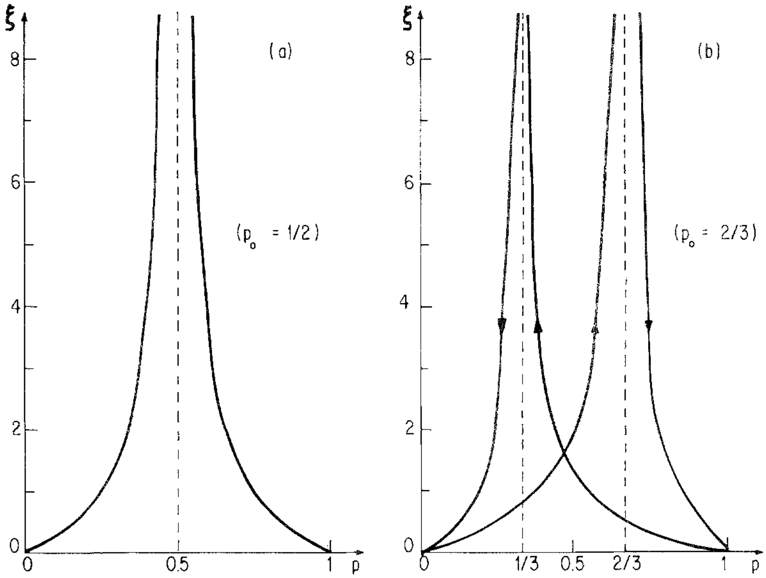


Fig. 1. The p dependence of the mean finite cluster size ξ for (a) $p_0 = 1/2$ (simple majority); (b) $p_0 = 2/3$ (the arrows refer to increasing or decreasing p).

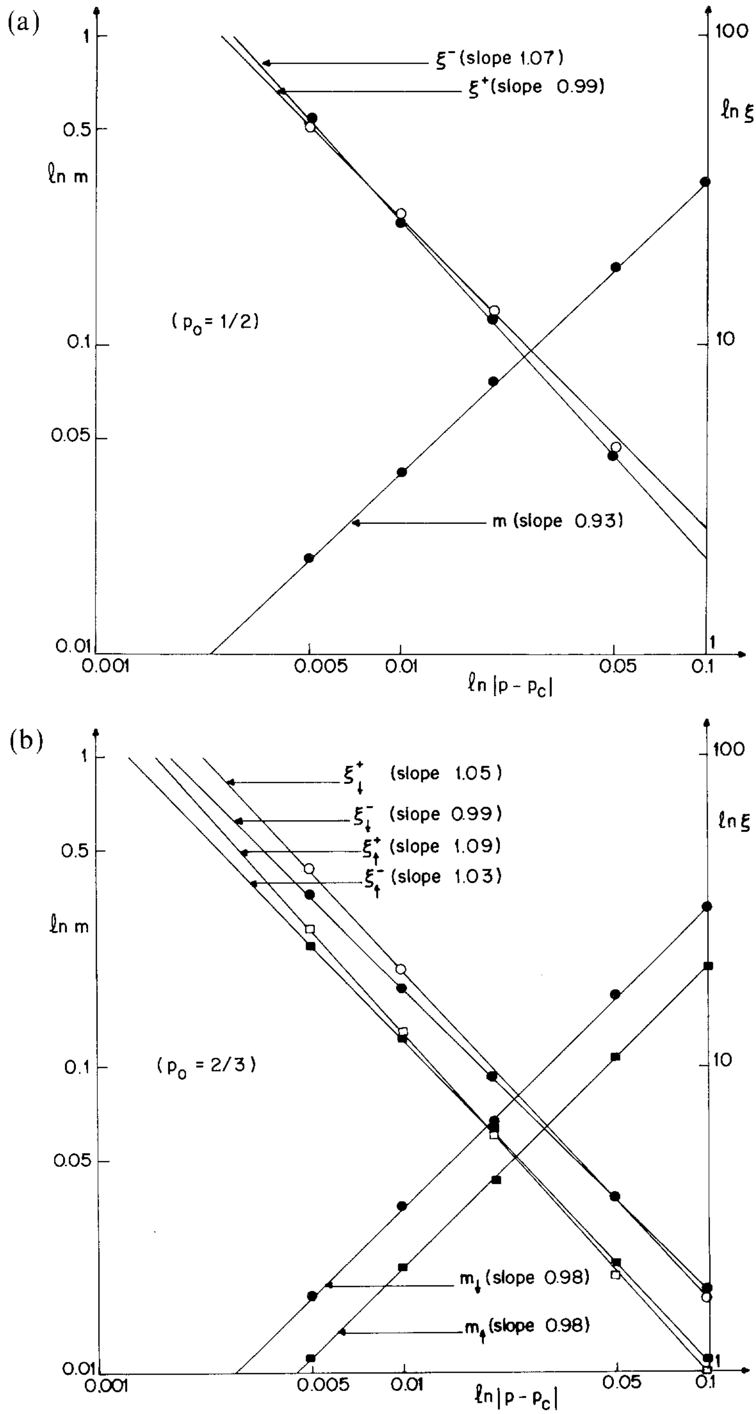
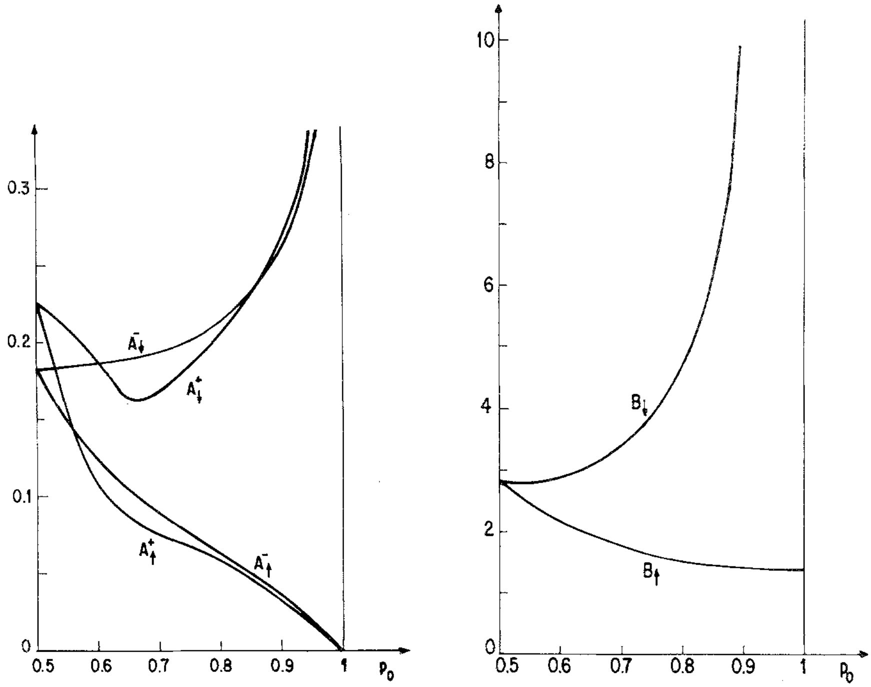


Fig. 2. The p-dependences of ξ and the order parameter m for (a) $p_0 = 1/2$; (b) $p_0 = 2/3$.



(a) (b)

Fig. 3. (a) The p_0 dependences of the length ξ critical amplitudes $A_\uparrow^+$, $A_\uparrow^-$, $A_\downarrow^+$, and $A_\downarrow^-$; (b) p_0 dependences of the order parameter critical amplitudes $B_\downarrow$ and $B_\uparrow$.

we have fixed p_0 and p (as well as N_0 and L), we randomly construct a strip configuration and choose its center. We then consider increasingly larger strip pieces around the center and check whether a black majority (p_0 majority, of course) is *maintained* until the size of the strip piece attains L (length of the entire strip): m is defined as the ratio of times a black majority succeeded to remain so *until the end.* In the $(L, N_0) \to (\infty, \infty)$ limit we expect that: (i) when p increases from 0 to p_0, m vanishes; (ii) when p increases from p_0 to 1, m increases from 0 to 1 $[m \sim B_\uparrow(p-p_0)^\beta$ in the neighborhood of $p_0]$; (iii) when p decreases from 1 to $1-p_0$, m decreases from 1 to 0 $\{m \sim B_\downarrow[p-(1-p_0)]^\beta\}$; (iv) when p decreases from $1-p_0$ to 0, m vanishes again. In Fig. 4 we present the results corresponding to $p_0 = 1/2$ and $p_0 = 2/3$. As illustrated in Fig. 2, β seems to be p_0 independent and equals 1 ± 0.1. In Fig. 3b we present the p_0 evolution of $B_\uparrow$ and $B_\downarrow$.

Let us conclude by noting that the hysteresis which occurs in this

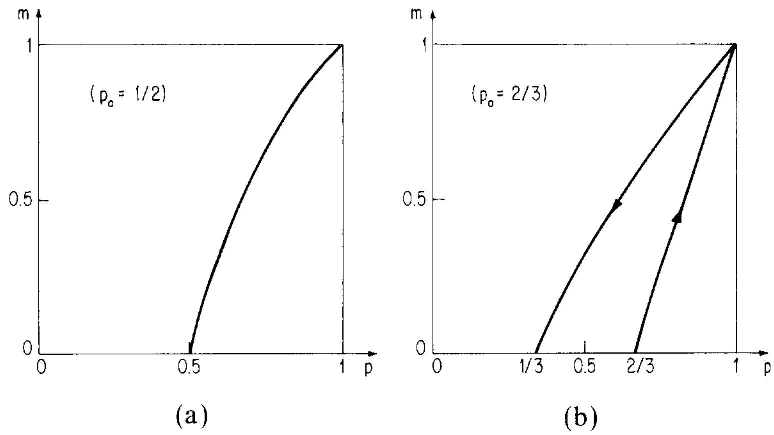


Fig. 4. The p dependence of the order parameter m for (a) $p_0 = 1/2$; (b) $p_0 = 2/3$ (the arrows refer to increasing or decreasing p).

model for $1/2 < p_0 < 1$ does not yield a *discontinuity* in the order parameter (at a *single* p_c for both increasing and decreasing p), contrarily to what occurs in standard first-order phase transitions. This is related to the fact that, in the present model, no conservation of "thermodynamic energy" is to be satisfied at the critical point(s). Let us mention at this point that the fact that the model remains invariant through interchange of the black and white colors implies that the (increasing p) p_0-majority model is isomorphic to the (decreasing p) $(1 - p_0)$-majority model.

Applications of this model are conceivable in a great variety of situations (e.g., at group, institutional, political levels) in which decisions are taken through voting (simple majority, 2/3 majority, etc); see ref. 6 for a discussion of "political" implications of a model which essentially reproduces that introduced in ref. 1 and extended here. Concrete examples of use would be very welcome. A further improvement in the sense of making the model more realistic would be to allow correlations in the voting process of the model (a new universality class could in principle emerge).

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